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Extremal graph characterization from the upper bound of the Laplacian spectral radius of weighted graphs ☆

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Abstract

We consider weighted graphs, where the edge weights are positive definite matrices. In this paper, we obtain two upper bounds on the spectral radius of the Laplacian matrix of weighted graphs and characterize graphs for which the bounds are attained. Moreover, we show that some known upper bounds on the Laplacian spectral radius of weighted and unweighted graphs can be deduced from our upper bounds.

AMS classification: 05C50

Keywords: Weighted graph; Laplacian matrix; Spectral radius; Upper bound

1. Introduction

We only consider undirected graphs which have no loops or multiple edges. Let $G = (V, E)$ be a connected graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. A weighted graph is a graph in which each edge is assigned a weight, which is usually positive number. An unweighted graph, or simply a graph, is thus a weighted graph with each of the edges bearing weight 1. Let d_i be the degree of the vertex i for $i = 1, 2, \ldots, n$. For $i \in V$, the set of neighbors of i and the average of the degrees of the vertices adjacent to i are denoted by N_i and m_i respectively. Let $|X|$ denote the cardinality of a finite set X.

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A weighted graph is a graph, each edge of which has been assigned a square matrix, called the weight of the edge. All the weight matrices will be assumed to be of the same order and will be assumed to be positive definite. In this paper, by “weighted graph” we will mean a “weighted graph with each of its edges bearing a positive definite matrix as weight”, unless otherwise stated. We now introduce some more notation. Let G be a weighted graph on n vertices. Denote by w_{ij} the positive definite weight matrix of order p of the edge ij, and assume that $w_{ij} = w_{ji}$. We write ij if vertices i and j are adjacent. Let $w_i = \sum_{j \in N_i} w_{ij}$, and we think of w_i as the weight matrix of the vertex i.

The Laplacian matrix of a graph G is a block matrix, denoted and defined as $L(G) = (l_{ij})$, where

$$l_{ij} = \begin{cases} w_i & \text{if } i = j, \\ -w_{ij} & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in the definition above, the zero denotes the $p \times p$ zero matrix. Thus $L(G)$ is a square matrix of order np. For any symmetric matrix B, let $\lambda_1(B)$ denote the largest eigenvalue of B. We set $\lambda_1 = \lambda_1(L(G))$.

Bipartite semiregular graph. A graph G is called a bipartite graph if G has no cycles of odd length; the vertex set $V(G)$ can be partitioned into two sets U and W in such a way that every edge in $E(G)$ connects a vertex in U with a vertex in W. A bipartite weighted graph G with a bipartition U and W such that every vertex i in U has the same $\lambda_1(w_i)$ and every vertex j in W has the same $\lambda_1(w_j)$, then G will be called a $(\lambda_1(w_i), \lambda_1(w_j))$-semiregular bipartite graph.

Regular graph. If every vertex i in V of weighted graph G has the same largest eigenvalue $\lambda_1(w_i)$, then G will be called a $\lambda_1(w_i)$-regular graph.

There exists a vast literature that studies the Laplacian eigenvalues and their relation to various properties on unweighted graphs. We refer the reader to [1,2,4–6,9–15] for surveys and more information. In [7], an upper bound on the largest Laplacian eigenvalue for weighted graphs was determined, where the edge weights are positive definite matrices.

The terminology and notation not defined may be found in [7]. The rest of this paper is organized as follows. In Section 2, we give Lemmas and recall some known results. These results are used, in Section 3, to provide proofs of Theorems 3.1 and 3.7. Theorems 3.1 and 3.7 give two upper bounds on the largest Laplacian eigenvalue for weighted graphs, where the edge weights are positive definite matrices. We also characterize graphs which achieve the upper bound. The results clearly generalize some known results for weighted and unweighted graphs.

2. Lemmas and results

Lemma 2.1 ((Rayleigh–Ritz) [16]). *If A is a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ then for any $\bar{x} \in R^n (\bar{x} \neq \bar{0})$*

$$\bar{x}^{\mathrm{T}} A \bar{x} \geqslant \lambda_n \bar{x}^{\mathrm{T}} \bar{x}. \tag{1}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of A corresponding to the least eigenvalue λ_n.

The following is a consequence of the Cauchy–Schwarz inequality.

Lemma 2.2. *If A is a real symmetric positive definite $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, then for any $\bar{x} \in R^n$ $(\bar{x} \neq \bar{0})$, $\bar{y} \in R^n (\bar{y} \neq \bar{0})$*

$$|\bar{x}^{\mathrm{T}} A \bar{y}| \leqslant \lambda_1 \sqrt{\bar{x}^{\mathrm{T}}\bar{x}}\sqrt{\bar{y}^{\mathrm{T}}\bar{y}}. \tag{2}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of A corresponding to the largest eigenvalue λ_1 and $\bar{y} = \alpha\bar{x}$ for some $\alpha \in R$.

Lemma 2.3. *Let G be a weighted graph, and let w_{ij} be the positive definite weight matrix of order p of the edge ij. Also let $\bar{x}$ be an eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j. Then*

$$\lambda_1(w_i) = \sum_{j \in N_i} \lambda_1(w_{ij}), \tag{3}$$

where $w_i = \sum_{j \in N_i} w_{ij}$.

Proof. Since $\bar{x}$ is a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j; we have

$$w_i\bar{x} = \sum_{j \in N_i} w_{ij}\bar{x} = \sum_{j \in N_i} \lambda_1(w_{ij})\bar{x}.$$

Thus $\sum_{j \in N_i} \lambda_1(w_{ij})$ is an eigenvalue of w_i. So, $\sum_{j \in N_i} \lambda_1(w_{ij}) \leqslant \lambda_1(w_i)$.

But we have $\lambda_1(w_i) \leqslant \sum_{j \in N_i} \lambda_1(w_{ij})$, as w_{ij}'s are positive definite matrices.

Using these two results, we get the required result (3). $\square$

The following Lemma 2.4 for unweighted graph has been proved in [3]. Here we include a detailed proof of the generalize result.

Lemma 2.4. *Let G be a simple connected weighted graph and let $\bar{x}$ be a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j. Also let $\gamma_i = \lambda_1(w_i)$ and $\bar{\gamma}_i = \frac{\sum_{k \in N_i} \lambda_1(w_{ik})\lambda_1(w_k)}{\lambda_1(w_i)}$, for all $i \in V$. Then $\gamma_1 + \bar{\gamma}_1 = \gamma_2 + \bar{\gamma}_2 = \cdots = \gamma_n + \bar{\gamma}_n$ holds if and only if G is a regular graph or G is a bipartite semiregular graph.*

Proof. Since $\bar{x}$ is a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j; then by Lemma 2.3 we have

$$\gamma_i = \lambda_1(w_i) = \sum_{k \in N_i} \lambda_1(w_{ik}) \quad \text{for all } i.$$

If G is a regular graph or G is a bipartite semiregular graph then $\gamma_1 + \bar{\gamma}_1 = \gamma_2 + \bar{\gamma}_2 = \cdots = \gamma_n + \bar{\gamma}_n$ holds.

Conversely, let $\gamma_1 + \bar{\gamma}_1 = \gamma_2 + \bar{\gamma}_2 = \cdots = \gamma_n + \bar{\gamma}_n$. Now we have to prove that G is a regular graph or G is a bipartite semiregular graph.

Let us take a vertex n of degree r where $\gamma_n \leqslant \gamma_k$ for all $k \in V$ and $\gamma_n = \alpha$, say. Further let vertex n be adjacent to the vertices $i_1, i_2, \ldots, i_r$. If possible, let $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_r}$ be not all of them equal and among them let γ_{i_1} be the highest. For vertex n, $\gamma_n + \bar{\gamma}_n < \alpha + \gamma_{i_1}$ and for vertex i_1, $\gamma_{i_1} + \bar{\gamma}_{i_1} \geqslant \gamma_{i_1} + \alpha$. From these two relations, we get $\gamma_n + \bar{\gamma}_n < \gamma_{i_1} + \bar{\gamma}_{i_1}$, a contradiction. Therefore we can conclude that $\gamma_{i_1} = \gamma_{i_2} = \cdots = \gamma_{i_r} = \beta$, say.

Let vertex i_1 be adjacent to the vertices n, $i_{r+1}, i_{r+2}, \ldots, i_{r+s-1}$, where s is the degree of vertex i_1. Since γ_n is the least, then $\gamma_k \geqslant \gamma_n$, $k = i_{r+1}, i_{r+2}, \ldots, i_{r+s-1}$. If possible, let one of γ_k, $k = i_{r+1}, i_{r+2}, \ldots, i_{r+s-1}$ be greater than α. Then for vertex i_1, $\gamma_{i_1} + \bar{\gamma}_{i_1} > \gamma_{i_1} + \alpha = \gamma_n + \bar{\gamma}_n$.

Again this is not possible. Therefore $\gamma_k = \alpha, k = i_{r+1}, i_{r+2}, \ldots, i_{r+s-1}$. Similarly we have $\gamma_k = \alpha, k \in N_t, t = i_2, i_3, \ldots, i_r$. Continuing the procedure, we can show that the vertices having the same largest eigenvalue α are adjacent to the vertices those have the same largest eigenvalue β and vice versa.

First we assume that G is bipartite. In this case G is a bipartite semiregular graph.

Next we assume that G is not bipartite. Then G has an odd cycle $jj_1j_2\ldots j_{k-1}j_k\acute{j}_k\acute{j}_{k-1}\ldots\acute{j}_2\acute{j}_1 j$. Therefore the largest eigenvalue of the vertex j is either α or β. Without loss of generality, we can assume that α is the largest eigenvalue of vertex j. Therefore j_1 and $\acute{j}_1$ both have the largest eigenvalue β. Similarly we can say that j_k and $\acute{j}_k$ both have the largest eigenvalue α if k is even, or β if k is odd. But both j_k and $\acute{j}_k$ are adjacent vertices, so α and β must be equal. Hence G is a regular graph. □

Lemma 2.5 [7]. *Let G be a $(\lambda_1(w_i), \lambda_1(w_j))$-semiregular bipartite graph of order n such that the first l vertices of the same largest eigenvalue $\lambda_1(w_i)$ and the remaining m vertices of the same largest eigenvalue $\lambda_1(w_j)$. Also let $\bar{x}$ be a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j; where $w_i = \sum_{k\in N_i} w_{ik}$, for all i. Then $\lambda_1(w_i) + \lambda_1(w_j)$ is the largest eigenvalue of $L(G)$ and the corresponding eigenvector is*

$$(\underbrace{\lambda_1(w_i)\bar{x}^{\mathrm{T}}, \lambda_1(w_i)\bar{x}^{\mathrm{T}}, \ldots, \lambda_1(w_i)\bar{x}^{\mathrm{T}}}_{l}, \underbrace{-\lambda_1(w_j)\bar{x}^{\mathrm{T}}, -\lambda_1(w_j)\bar{x}^{\mathrm{T}}, \ldots, -\lambda_1(w_j)\bar{x}^{\mathrm{T}}}_{m})^{\mathrm{T}}.$$

Lemma 2.6 [8]. *Let G be a weighted graph of order n. Then*

$$\lambda_1 > \max_i \lambda_1(w_i),$$

where w_i is the weight matrix of the ith vertex of graph G.

3. Main result

In this section we find two upper bounds on the largest Laplacian eigenvalue and characterize the graphs for which the largest Laplacian eigenvalue is equal to the upper bound.

Theorem 3.1. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leqslant \max_{ij}\left\{\sqrt{\sum_{k\in N_i}\lambda_1(w_{ik})\left(\sum_{r\in N_i}\lambda_1(w_{ir})+\sum_{s\in N_k}\lambda_1(w_{ks})\right)+\sum_{k\in N_j}\lambda_1(w_{jk})\left(\sum_{r\in N_j}\lambda_1(w_{jr})+\sum_{s\in N_k}\lambda_1(w_{ks})\right)}\right\}, \tag{4}$$

where w_{ij} is the positive definite weight matrix of order p of the edge $ij \in E$. Moreover equality holds in (4) if and only if

(i) *G is a bipartite semiregular graph; and*
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.*

Proof. Let $\bar{\mathbf{X}} = (\bar{x}_1^{\mathrm{T}}, \bar{x}_2^{\mathrm{T}}, \ldots, \bar{x}_n^{\mathrm{T}})^{\mathrm{T}}$ be an eigenvector corresponding to the largest eigenvalue λ_1 of $L(G)$. We assume that vector $\bar{x}_{i_0j_0}$ $(\bar{x}_{i_0j_0} = \bar{x}_{i_0} - \bar{x}_{j_0})$ corresponding to the edge i_0j_0 such that $\bar{x}_{i_0j_0}^{\mathrm{T}}\bar{x}_{i_0j_0} = \max_{rs}\{\bar{x}_{rs}^{\mathrm{T}}\bar{x}_{rs}\}$.

The $(i,\ j)$th element of $L(G)$ is

$$\begin{cases} w_i & \text{if } i=j, \\ -w_{ij} & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$L(G)\bar{\mathbf{X}} = \lambda_1 \bar{\mathbf{X}}. \tag{5}$$

From the i_0th equation of (5), we have

$$\lambda_1 \bar{x}_{i_0} = w_{i_0}\bar{x}_{i_0} - \sum_{k\in N_{i_0}} w_{i_0k}\bar{x}_k,$$

$$\text{i.e., } \lambda_1 \bar{x}_{i_0} = \sum_{k\in N_{i_0}} w_{i_0k}\bar{x}_{i_0} - \sum_{k\in N_{i_0}} w_{i_0k}\bar{x}_k,$$

$$\text{i.e., } \lambda_1 \bar{x}_{i_0} = \sum_{k\in N_{i_0}} w_{i_0k}\bar{x}_{i_0k}, \quad \bar{x}_{i_0k} = \bar{x}_{i_0} - \bar{x}_k \quad \text{(say)}. \tag{6}$$

From the j_0th equation of (5), we have

$$\lambda_1 \bar{x}_{j_0} = \sum_{k\in N_{j_0}} w_{j_0k}\bar{x}_{j_0k}, \quad \bar{x}_{j_0k} = \bar{x}_{j_0} - \bar{x}_k \quad \text{(say)}. \tag{7}$$

Subtracting (7) from (6), we get

$$\lambda_1 \bar{x}_{i_0j_0} = \sum_{k\in N_{i_0}} w_{i_0k}\bar{x}_{i_0k} - \sum_{k\in N_{j_0}} w_{j_0k}\bar{x}_{j_0k}, \quad \bar{x}_{i_0j_0} = \bar{x}_{i_0} - \bar{x}_{j_0}. \tag{8}$$

Multiplying both sides by $\bar{x}^{\mathrm{T}}_{i_0j_0}$ to the left of the above equation, we get

$$\begin{aligned} \lambda_1 \bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0} &= \sum_{k\in N_{i_0}} \bar{x}^{\mathrm{T}}_{i_0j_0} w_{i_0k}\bar{x}_{i_0k} - \sum_{k\in N_{j_0}} \bar{x}^{\mathrm{T}}_{i_0j_0} w_{j_0k}\bar{x}_{j_0k} \\ &\leqslant \left| \sum_{k\in N_{i_0}} \bar{x}^{\mathrm{T}}_{i_0j_0} w_{i_0k}\bar{x}_{i_0k} - \sum_{k\in N_{j_0}} \bar{x}^{\mathrm{T}}_{i_0j_0} w_{j_0k}\bar{x}_{j_0k} \right| && (9) \\ &\leqslant \sum_{k\in N_{i_0}} \left|\bar{x}^{\mathrm{T}}_{i_0j_0} w_{i_0k}\bar{x}_{i_0k}\right| + \sum_{k\in N_{j_0}} \left|\bar{x}^{\mathrm{T}}_{i_0j_0} w_{j_0k}\bar{x}_{j_0k}\right| && (10) \\ &\leqslant \sum_{k\in N_{i_0}} \lambda_1(w_{i_0k})\sqrt{\bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}}\sqrt{\bar{x}^{\mathrm{T}}_{i_0k}\bar{x}_{i_0k}} \\ &\quad + \sum_{k\in N_{j_0}} \lambda_1(w_{j_0k})\sqrt{\bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}}\sqrt{\bar{x}^{\mathrm{T}}_{j_0k}\bar{x}_{j_0k}}. && (11) \end{aligned}$$

If $\bar{x}_1 = \bar{x}_2 = \cdots = \bar{x}_n$, then from (6) we get

$$\lambda_1 \bar{x}_{i_0} = \bar{0},$$
$$\text{i.e., } \bar{x}_{i_0} = \bar{0}\ (\lambda_1 \neq 0), \text{ a contradiction as } \bar{\mathbf{X}} \text{ is not equal to } \bar{0}.$$

Using the above result we conclude that $\bar{x}_{i_0j_0}$ is a non-zero vector as $\bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}$ is maximum and $\bar{\mathbf{X}}$ is nonzero. From (11), we get

$$\lambda_1\sqrt{\bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}} \leqslant \sum_{k\in N_{i_0}} \lambda_1(w_{i_0k})\sqrt{\bar{x}^{\mathrm{T}}_{i_0k}\bar{x}_{i_0k}} + \sum_{k\in N_{j_0}} \lambda_1(w_{j_0k})\sqrt{\bar{x}^{\mathrm{T}}_{j_0k}\bar{x}_{j_0k}}. \tag{12}$$

So we have for $i_0k \in E$ and $\bar{x}^{\mathrm{T}}_{i_0k}\bar{x}_{i_0k} \neq 0$,

$$\begin{aligned}\lambda_1\sqrt{\bar{x}^{\mathrm{T}}_{i_0k}\bar{x}_{i_0k}} &\leqslant \sum_{r\in N_{i_0}} \lambda_1(w_{i_0r})\sqrt{\bar{x}^{\mathrm{T}}_{i_0r}\bar{x}_{i_0r}} + \sum_{r\in N_k} \lambda_1(w_{kr})\sqrt{\bar{x}^{\mathrm{T}}_{kr}\bar{x}_{kr}} \\ &\leqslant \left(\sum_{r\in N_{i_0}} \lambda_1(w_{i_0r}) + \sum_{r\in N_k} \lambda_1(w_{kr})\right)\sqrt{\bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}},\ \text{as } \bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0} \text{ is maximum.}\end{aligned} \tag{13}$$

Similarly we have for $j_0k \in E$ and $\bar{x}^{\mathrm{T}}_{j_0k}\bar{x}_{j_0k} \neq 0$,

$$\lambda_1\sqrt{\bar{x}^{\mathrm{T}}_{j_0k}\bar{x}_{j_0k}} \leqslant \left(\sum_{r\in N_{j_0}} \lambda_1(w_{j_0r}) + \sum_{r\in N_k} \lambda_1(w_{kr})\right)\sqrt{\bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}}. \tag{14}$$

Multiplying both sides of (12) by λ_1 and using relations (13) and (14), we get

$$\lambda_1 \leqslant \sqrt{\begin{aligned}&\sum_{k\in N_{i_0}} \lambda_1(w_{i_0k})\left(\sum_{r\in N_{i_0}} \lambda_1(w_{i_0r}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\\ &\quad+\sum_{k\in N_{j_0}} \lambda_1(w_{j_0k})\left(\sum_{r\in N_{j_0}} \lambda_1(w_{j_0r}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\end{aligned}}\quad i_0j_0 \in E. \tag{15}$$

Thus, we complete the first part of the proof.

Now suppose that equality holds in (4). Then all inequalities in the above argument must be equalities.

If there exists at least one k such that $k \in N_{i_0}$ and $\bar{x}_{i_0k} = \bar{0}$; then from (12) and (15) we have

$$\lambda_1 < \sqrt{\begin{aligned}&\sum_{k\in N_{i_0}} \lambda_1(w_{i_0k})\left(\sum_{r\in N_{i_0}} \lambda_1(w_{i_0r}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\\ &\quad+\sum_{k\in N_{j_0}} \lambda_1(w_{j_0k})\left(\sum_{r\in N_{j_0}} \lambda_1(w_{j_0r}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\end{aligned}}\quad i_0j_0 \in E,$$

a contradiction as equality holds in (15).

Therefore $\bar{x}_{i_0k} \neq \bar{0}$, $k \in N_{i_0}$. Similarly we can show that $\bar{x}_{j_0k} \neq \bar{0}$, $k \in N_{j_0}$.

From equality in (11) and using Lemma 2.2, we get that $\bar{x}_{i_0k}$ is an eigenvector of w_{i_0k} for the largest eigenvalue $\lambda_1(w_{i_0k})$, $k \in N_{i_0}$ and for any $k \in N_{i_0}$, $\bar{x}_{i_0k} = b_{i_0k}\bar{x}_{i_0j_0}$, for some b_{i_0k}. Also we get that $\bar{x}_{j_0k}$ is an eigenvector of w_{j_0k} for the largest eigenvalue $\lambda_1(w_{j_0k})$, $k \in N_{j_0}$ and for any $k \in N_{j_0}$, $\bar{x}_{j_0k} = c_{j_0k}\bar{x}_{i_0j_0}$, for some c_{j_0k}.

From equality in (13), we get

$$\begin{aligned}&\bar{x}^{\mathrm{T}}_{i_0k}\bar{x}_{i_0k} = \bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}, \quad k \in N_{i_0},\\ &\text{and}\quad \bar{x}^{\mathrm{T}}_{kr}\bar{x}_{kr} = \bar{x}^{\mathrm{T}}_{i_0j_0}\bar{x}_{i_0j_0}, \quad r \in N_k,\ \ k \in N_{i_0}.\end{aligned} \tag{16}$$

From these two results we have

$$\begin{aligned}&b^2_{i_0k} = 1, \quad k \in N_{i_0} \quad \text{and} \quad c^2_{j_0k} = 1,\ \ k \in N_{j_0} \text{ as } \bar{x}_{i_0j_0} \text{ is non zero,}\\ &\text{i.e., } b_{i_0k} = \pm 1,\ \ k \in N_{i_0} \quad \text{and} \quad c_{j_0k} = \pm 1,\ \ k \in N_{j_0}.\end{aligned} \tag{17}$$

Since w_{i_0k} is a positive definite matrix and $\bar{x}_{i_0j_0}$ is an eigenvector of w_{i_0k} for the largest eigenvalue $\lambda_1(w_{i_0k})$, we have

$$\bar{x}_{i_0j_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0j_0} > 0, \tag{18}$$

From equality in (9) and (10), we get

$$\begin{aligned}
&\sum_{k\in N_{i_0}} b_{i_0k}\bar{x}_{i_0j_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0j_0} - \sum_{k\in N_{j_0}} c_{j_0k}\bar{x}_{i_0j_0}^{\mathrm{T}} w_{j_0k}\bar{x}_{i_0j_0}\\
&\quad= \sum_{k\in N_{i_0}} |b_{i_0k}||\bar{x}_{i_0j_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0j_0}| + \sum_{k\in N_{j_0}} |c_{j_0k}||\bar{x}_{i_0j_0}^{\mathrm{T}} w_{j_0k}\bar{x}_{i_0j_0}|,\\
&\text{i.e., } \sum_{k\in N_{i_0}} (|b_{i_0k}| - b_{i_0k})\bar{x}_{i_0j_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0j_0} + \sum_{k\in N_{j_0}} (|c_{j_0k}| + c_{j_0k})\bar{x}_{i_0j_0}^{\mathrm{T}} w_{j_0k}\bar{x}_{i_0j_0}\\
&\quad= 0 \text{ by (18)},\\
&\text{i.e., } |b_{i_0k}| - b_{i_0k} = 0,\ \ k\in N_{i_0} \quad\text{and}\quad |c_{j_0k}| + c_{j_0k} = 0,\ \ k\in N_{j_0}, \quad \text{by (18)},\\
&\text{i.e., } b_{i_0k} = 1,\ \ k\in N_{i_0} \text{ and } c_{j_0k} = -1,\ \ k\in N_{j_0},\ \text{as } b_{i_0k} = \pm 1 \text{ and } c_{j_0k} = \pm 1.
\end{aligned}$$

Thus $\bar{x}_{i_0k} = \bar{x}_{i_0j_0}, k \in N_{i_0}$ and $\bar{x}_{j_0k} = -\bar{x}_{i_0j_0}$, $k \in N_{j_0}$, that is, $\bar{x}_k = \bar{x}_{j_0}$, $k \in N_{i_0}$ and $\bar{x}_k = \bar{x}_{i_0}$, $k \in N_{j_0}$.

For each $k \in N_{i_0}$ and $r \in N_k, \bar{x}_r = \bar{x}_{i_0}$, and for each $k \in N_{j_0}$ and $r \in N_k, \bar{x}_r = \bar{x}_{j_0}$. Let $U = \{k : \bar{x}_k = \bar{x}_{i_0}\}$ and $W = \{k : \bar{x}_k = \bar{x}_{j_0}\}$. One can find that $U \neq \emptyset \neq W$ and $U \cap W = \emptyset$ as $\bar{x}_{i_0} \neq \bar{x}_{j_0}$. By a similar discussion and the fact that G is connected, each vertex of G has value $\bar{x}_{i_0}$ or $\bar{x}_{j_0}$. So $V(G) = U \cup W$ and G is bipartite with bipartition (U, W). Moreover, the w_{ij}'s have a common eigenvector $(\bar{x}_{i_0j_0})$ corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.

For $i \in U$,

$$\begin{aligned}
\lambda_1\bar{x}_{i_0} &= w_i\bar{x}_{i_0} - \sum_{r\in N_i} w_{ir}\bar{x}_r\\
&= w_i\bar{x}_{i_0} - \sum_{r\in N_i} w_{ir}\bar{x}_{j_0} \quad \text{as } \bar{x}_r = \bar{x}_{j_0}, r\in N_i\\
&= w_i\bar{x}_{i_0} - w_i\bar{x}_{j_0}\\
&= \sum_{r\in N_i} w_{ir}\bar{x}_{i_0j_0} \quad \text{as } \bar{x}_{i_0j_0} = \bar{x}_{i_0} - \bar{x}_{j_0}\\
&= \sum_{r\in N_i} \lambda_1(w_{ir})\bar{x}_{i_0j_0}\\
&= \lambda_1(w_i)\bar{x}_{i_0j_0} \quad \text{by Lemma 2.3.}
\end{aligned}$$

For $i, k \in U$,

$$\begin{aligned}
&\lambda_1\bar{x}_{i_0} = \lambda_1(w_i)\bar{x}_{i_0j_0} = \lambda_1(w_k)\bar{x}_{i_0j_0},\\
&\text{i.e., } (\lambda_1(w_i) - \lambda_1(w_k))\bar{x}_{i_0j_0} = \bar{0},\\
&\text{i.e., } \lambda_1(w_i) = \lambda_1(w_k) \quad \text{as } \bar{x}_{i_0j_0} \text{ is nonzero.}
\end{aligned} \tag{19}$$

Therefore $\lambda_1(w_i)$ is constant for all $i \in U$. Similarly we can show that $\lambda_1(w_j)$ is constant for all $j \in W$. Hence G is a bipartite semiregular graph.

Conversely, suppose that conditions (i)–(ii) of the theorem hold for the graph G. Let G be a $(\lambda_1(w_{i_0}), \lambda_1(w_{j_0}))$-semiregular bipartite graph. Using Lemma 2.5 we have

$$\lambda_1 = \lambda_1(w_{i_0}) + \lambda_1(w_{j_0}).$$

Since the w_{ij}'s have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j; using Lemma 2.3 we get

$$\lambda_1(w_i) = \sum_{r\in N_i} \lambda_1(w_{ir}) \quad \text{for all } i.$$

Now, for $ij \in E$,

$$\begin{aligned}&\sum_{k\in N_i} \lambda_1(w_{ik}) \left(\sum_{r\in N_i} \lambda_1(w_{ir}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\\ &\quad+ \sum_{k\in N_j} \lambda_1(w_{jk}) \left(\sum_{r\in N_j} \lambda_1(w_{jr}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\\ &\quad= (\lambda_1(w_{i_0}) + \lambda_1(w_{j_0}))^2.\end{aligned}$$

Hence the theorem is proved. □

Corollary 3.2. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leqslant \max_i \left\{ \sqrt{2\sum_{k\in N_i} \lambda_1(w_{ik})\left(\sum_{r\in N_i} \lambda_1(w_{ir}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)} \right\}, \tag{20}$$

where w_{ij} is the positive definite weight matrix of order p of the edge $ij \in E$. Moreover equality holds in (20) if and only if

(i) *G is a bipartite regular graph*;
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.*

Proof. We can see easily that

$$\begin{aligned}&\max_{ij} \left\{ \sqrt{\begin{aligned}&\sum_{k\in N_i} \lambda_1(w_{ik}) \left(\sum_{r\in N_i} \lambda_1(w_{ir}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\\ &\quad+ \sum_{k\in N_j} \lambda_1(w_{jk}) \left(\sum_{r\in N_j} \lambda_1(w_{jr}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)\end{aligned}} \right\}\\ &\quad\leqslant \max_i \left\{ \sqrt{2\sum_{k\in N_i} \lambda_1(w_{ik}) \left(\sum_{r\in N_i} \lambda_1(w_{ir}) + \sum_{s\in N_k} \lambda_1(w_{ks})\right)} \right\}.\end{aligned}$$

Using Theorem 3.1 we get the required result. □

Corollary 3.3. *Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then*

$$\lambda_1 \leqslant \max_i \left\{ \sqrt{2w_i(w_i + \bar{w}_i)} \right\},$$

where $\bar{w}_i = \frac{\sum_{k\in N_i} w_{ik} w_k}{w_i}$ and w_i is the weight of vertex i. Moreover equality holds if and only if G is a bipartite regular graph.

Proof. For weighted graph where weight w_{ij} is a positive number, we have $\lambda_1(w_i) = w_i$ and $\lambda_1(w_{ij}) = w_{ij}$ for all i, j. Using Corollary 3.2 we get the required result. □

Corollary 3.4 [12]. *Let G be a simple connected unweighted graph. Then*

$$\lambda_1 \leqslant \max_i \left\{\sqrt{2d_i(d_i+m_i)}\right\}, \tag{21}$$

where d_i, m_i are the degree of vertex i and the average degrees of the adjacent vertices of vertex i, respectively. Moreover equality holds in (21) if and only if G is a bipartite regular graph.

Proof. For unweighted graph, $w_{ij}=1$ and $w_i=d_i$. Using Corollary 3.3 we get the required result. □

Corollary 3.5. *Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then*

$$\lambda_1 \leqslant \max_{ij} \left\{\sqrt{w_i(w_i+\bar{w}_i)+w_j(w_j+\bar{w}_j)}\right\}, \tag{22}$$

where $\bar{w}_i = \frac{\sum_{k\in N_i} w_{ik}w_k}{w_i}$, and w_i is the weight of vertex i. Moreover equality holds in (22) if and only if G is a bipartite semiregular graph.

Proof. We have $\lambda_1(w_i)=w_i$ and $\lambda_1(w_{ij})=w_{ij}$ for all i,j. Using Theorem 3.1 we get the required result. □

Corollary 3.6 [17]. *Let G be a simple connected unweighted graph. Then*

$$\lambda_1 \leqslant \max_{ij} \left\{\sqrt{d_i(d_i+m_i)+d_j(d_j+m_j)}\right\}, \tag{23}$$

where d_i is the degree of vertex i and m_i is the average of the degrees of the vertices adjacent to vertex i. Moreover equality holds in (23) if and only if G is a bipartite semiregular graph.

Proof. The proof follows directly from Corollary 3.5. □

Theorem 3.7. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leqslant \max_{ij} \left\{\frac{\lambda_1(w_i)+\lambda_1(w_j)+\sqrt{(\lambda_1(w_i)-\lambda_1(w_j))^2+4\bar{\gamma}_i\bar{\gamma}_j}}{2}\right\}, \tag{24}$$

where $\bar{\gamma}_i = \frac{\sum_{k\in N_i} \lambda_1(w_{ik})\lambda_1(w_k)}{\lambda_1(w_i)}$, and w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (24) if and only if

(i) *G is a bipartite regular graph or G is a bipartite semiregular graph; and*
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i,j.*

Proof. Let $\bar{\mathbf{X}} = (\bar{x}_1^{\mathrm{T}}, \bar{x}_2^{\mathrm{T}}, \ldots, \bar{x}_n^{\mathrm{T}})^{\mathrm{T}}$ be an eigenvector corresponding to the largest eigenvalue λ_1 of $M(G)^{-1}L(G)M(G)$, where $M(G)$ is the block diagonal matrix diag $(\gamma_1 I_{p,p}, \gamma_2 I_{p,p}, \ldots, \gamma_n I_{p,p})$, $\gamma_i=\lambda_1(w_i)$, $i=1,2,\ldots,n$. We assume that $\bar{x}_{i_0}$ is the vector component of $\bar{\mathbf{X}}$ such that $\bar{x}_{i_0}^{\mathrm{T}}\bar{x}_{i_0} = \max_k\{\bar{x}_k^{\mathrm{T}}\bar{x}_k\}$. Since $\bar{\mathbf{X}}$ is nonzero, so is $\bar{x}_{i_0}$.

$$\begin{aligned} &\text{Let} \quad \bar{x}_{j_0}^{\mathrm{T}}\bar{x}_{j_0} = \max_{k\in N_{i_0}}\{\bar{x}_k^{\mathrm{T}}\bar{x}_k\},\\ &\text{i.e.,} \quad \bar{x}_{j_0}^{\mathrm{T}}\bar{x}_{j_0} \geqslant \bar{x}_k^{\mathrm{T}}\bar{x}_k,\ k\in N_{i_0}. \end{aligned} \tag{25}$$

We have

$$\{M(G)^{-1}L(G)M(G)\}\bar{\mathbf{X}} = \lambda_1\bar{\mathbf{X}}. \tag{26}$$

From the i_0th equation of (26), we have

$$\begin{aligned}
\lambda_1\bar{x}_{i_0} &= w_{i_0}\bar{x}_{i_0} - \sum_{k\in N_{i_0}} \frac{\gamma_k w_{i_0k}}{\gamma_{i_0}}\bar{x}_k,\\
\text{i.e., } (\lambda_1 I_{p,p} - w_{i_0})\bar{x}_{i_0} &= -\sum_{k\in N_{i_0}} \frac{\gamma_k w_{i_0k}}{\gamma_{i_0}}\bar{x}_k,\\
\text{i.e., } \bar{x}_{i_0}^{\mathrm{T}}(\lambda_1 I_{p,p} - w_{i_0})\bar{x}_{i_0} &= -\sum_{k\in N_{i_0}} \frac{\gamma_k}{\gamma_{i_0}}\bar{x}_{i_0}^{\mathrm{T}} w_{i_0k}\bar{x}_k\\
&\leqslant \sum_{k\in N_{i_0}} \frac{\gamma_k}{\gamma_{i_0}}\left|\bar{x}_{i_0}^{\mathrm{T}} w_{i_0k}\bar{x}_k\right|, \qquad (27)\\
&\leqslant \sum_{k\in N_{i_0}} \frac{\gamma_k}{\gamma_{i_0}}\lambda_1(w_{i_0k})\sqrt{\bar{x}_{i_0}^{\mathrm{T}}\bar{x}_{i_0}}\sqrt{\bar{x}_k^{\mathrm{T}}\bar{x}_k}, \quad \text{by (2)}, \qquad (28)\\
&\leqslant \sqrt{\bar{x}_{i_0}^{\mathrm{T}}\bar{x}_{i_0}}\sqrt{\bar{x}_{j_0}^{\mathrm{T}}\bar{x}_{j_0}} \sum_{k\in N_{i_0}} \frac{\gamma_k}{\gamma_{i_0}}\lambda_1(w_{i_0k}), \quad \text{by (25)}. \qquad (29)
\end{aligned}$$

From (29), we get

$$(\lambda_1 - \lambda_1(w_{i_0}))\bar{x}_{i_0}^{\mathrm{T}}\bar{x}_{i_0} \leqslant \bar{x}_{i_0}^{\mathrm{T}}(\lambda_1 I_{p,p} - w_{i_0})\bar{x}_{i_0} \leqslant \bar{\gamma}_{i_0}\sqrt{\bar{x}_{i_0}^{\mathrm{T}}\bar{x}_{i_0}}\sqrt{\bar{x}_{j_0}^{\mathrm{T}}\bar{x}_{j_0}} \quad \text{by (1)}. \tag{30}$$

Similarly, from the j_0th equation of (26), we have

$$(\lambda_1 - \lambda_1(w_{j_0}))\bar{x}_{j_0}^{\mathrm{T}}\bar{x}_{j_0} \leqslant \bar{\gamma}_{j_0}\sqrt{\bar{x}_{i_0}^{\mathrm{T}}\bar{x}_{i_0}}\sqrt{\bar{x}_{j_0}^{\mathrm{T}}\bar{x}_{j_0}}. \tag{31}$$

From (30) and (31), we get

$$\begin{aligned}
&(\lambda_1 - \lambda_1(w_{i_0}))(\lambda_1 - \lambda_1(w_{j_0})) \leqslant \bar{\gamma}_{i_0}\bar{\gamma}_{j_0},\\
&\text{i.e., } \lambda_1^2 - (\lambda_1(w_{i_0}) + \lambda_1(w_{j_0}))\lambda_1 + \lambda_1(w_{i_0})\lambda_1(w_{j_0}) - \bar{\gamma}_{i_0}\bar{\gamma}_{j_0} \leqslant 0.
\end{aligned}$$

Thus

$$\lambda_1 \leqslant \frac{\lambda_1(w_{i_0}) + \lambda_1(w_{j_0}) + \sqrt{(\lambda_1(w_{i_0}) - \lambda_1(w_{j_0}))^2 + 4\bar{\gamma}_{i_0}\bar{\gamma}_{j_0}}}{2},$$

$$\text{where} \quad \bar{\gamma}_{i_0} = \frac{\sum_{k\in N_{i_0}} \lambda_1(w_{i_0k})\lambda_1(w_k)}{\lambda_1(w_{i_0})}.$$

This completes the proof of (24).

Now suppose that equality holds in (24). Then all inequalities in the above argument must be equalities.

If possible, let $\bar{x}_{j_0} = \bar{0}$. Then

$$\bar{x}_k = \bar{0}, \quad k \in N_{i_0}.$$

From the i_0th equation of (26), we get

$$\lambda_1\bar{x}_{i_0} = w_{i_0}\bar{x}_{i_0},$$

which implies that $\lambda_1 \leqslant \lambda_1(w_{i_0})$, is not possible by Lemma 2.6. Hence $\bar{x}_{j_0} \neq \bar{0}$.

From equality in (29), we get

$$\bar{x}_k^{\mathrm{T}}\bar{x}_k = \bar{x}_{j_0}^{\mathrm{T}}\bar{x}_{j_0}, \quad k \in N_{i_0}.$$

From this we get $\bar{x}_k \neq \bar{0}$, $k \in N_{i_0}$ as $\bar{x}_{j_0} \neq \bar{0}$.

From equality in (28) and using Lemma 2.2, we get that $\bar{x}_{i_0}$ is an eigenvector of w_{i_0k} for the largest eigenvalue $\lambda_1(w_{i_0k})$, $k \in N_{i_0}$ and for any $k \in N_{i_0}$, $\bar{x}_k = b_{i_0k}\bar{x}_{i_0}$, for some b_{i_0k}.

Since w_{i_0k} is a positive definite matrix and $\bar{x}_{i_0}$ is an eigenvector of w_{i_0k} for the largest eigenvalue $\lambda_1(w_{i_0k})$, we have

$$\bar{x}_{i_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0} > 0. \tag{32}$$

From equality in (27), we have

$$\begin{aligned}
&\textstyle\sum_{k\in N_{i_0}} \frac{\gamma_k}{\gamma_{i_0}}|b_{i_0k}\|\bar{x}_{i_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0}| = -\sum\limits_{k\in N_{i_0}} \frac{\gamma_k}{\gamma_{i_0}} b_{i_0k}(\bar{x}_{i_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0}), \quad \text{by } \bar{x}_k = b_{i_0k}\bar{x}_{i_0},\\
&\text{i.e., } \sum_{k\in N_{i_0}} (|b_{i_0k}| + b_{i_0k})\frac{\gamma_k}{\gamma_{i_0}}(\bar{x}_{i_0}^{\mathrm{T}} w_{i_0k}\bar{x}_{i_0}) = 0, \quad \text{by (32)},\\
&\text{i.e., } |b_{i_0k}| = -b_{i_0k}, k \in N_{i_0}, \quad \text{by (32)},\\
&\text{i.e., } b_{i_0k} < 0, \quad k \in N_{i_0},\\
&\text{i.e., } b_{i_0k} = -b \quad \text{for some } b, b > 0 \text{ (say)}.
\end{aligned}$$

Hence $\bar{x}_k = -b\bar{x}_{i_0}$, $k \in N_{i_0}$.

From equality in (30) and using Lemma 2.1, we get that $\bar{x}_{i_0}$ is a common eigenvector of w_{i_0} and w_{i_0k} for the largest eigenvalues respectively $\lambda_1(w_{i_0})$ and $\lambda_1(w_{i_0k})$, for all k, $k \in N_{i_0}$.

Similarly, from equality in (31), we get that $\bar{x}_{j_0}$ is a common eigenvector of w_{j_0} and w_{j_0k} for the largest eigenvalues respectively $\lambda_1(w_{j_0})$ and $\lambda_1(w_{j_0k})$, for all k, $k \in N_{j_0}$. Also, for any $k \in N_{j_0}$,

$$\bar{x}_k = -c\bar{x}_{j_0} \quad \text{for some } c > 0.$$

For $i_0 j_0 \in E$,

$$\bar{x}_{j_0} = -b\bar{x}_{i_0} \quad \text{and} \quad \bar{x}_{i_0} = -c\bar{x}_{j_0}.$$

From these relations, we get $bc = 1$, as $\bar{x}_{i_0}$ and $\bar{x}_{j_0}$ both are non-zero vector. Therefore $\bar{x}_k = -b\bar{x}_{i_0}$, $k \in N_{i_0}$ and $\bar{x}_k = \bar{x}_{i_0}$, $k \in N_{j_0}$.

Let $U = \{k : \bar{x}_k = \bar{x}_{i_0}\}$ and $W = \{k : \bar{x}_k = -b\bar{x}_{i_0}\}$. So, $N_{j_0} \subseteq U$ and $N_{i_0} \subseteq W$. Using the same technique as in Theorem 3.1 we conclude that G is bipartite. Moreover, $\bar{x}_{i_0}$ is a common eigenvector of w_i and w_{ij} corresponding to the largest eigenvalues $\lambda_1(w_i)$ and $\lambda_1(w_{ij})$ for all i, j.

From the ith equation of (26), we get

$$\lambda_1\lambda_1(w_i)\bar{x}_i = \lambda_1(w_i)w_i\bar{x}_i - \sum_{k\in N_i}\lambda_1(w_k)w_{ik}\bar{x}_k, \quad i = 1, 2, \ldots, n.$$

Taking summation on the above equation from $i = 1$ to n, we get

$$\begin{aligned}
&\lambda_1\sum_{i=1}^{n}\lambda_1(w_i)\bar{x}_i = 0,\\
&\text{i.e., } \sum_{k\in U}\lambda_1(w_k)\bar{x}_k + \sum_{k\in W}\lambda_1(w_k)\bar{x}_k = 0, \quad \text{as } \lambda_1 > 0,\\
&\text{i.e., } \sum_{k\in U}\lambda_1(w_k)\bar{x}_{i_0} - b\sum_{k\in W}\lambda_1(w_k)\bar{x}_{i_0} = 0,\\
&\text{i.e., } b = 1, \text{ as } \bar{x}_{i_0} \neq \bar{0} \quad \text{and} \quad \sum_{k\in U}\lambda_1(w_k) = \sum_{k\in W}\lambda_1(w_k).
\end{aligned}$$

So, $U = \{k : \bar{x}_k = \bar{x}_{i_0}\}$ and $W = \{k : \bar{x}_k = -\bar{x}_{i_0}\}$. For any i,

$$\lambda_1 \bar{x}_{i_0} = w_i \bar{x}_{i_0} + \sum_{k \in N_i} \frac{\gamma_k w_{ik}}{\gamma_i} \bar{x}_{i_0}$$

$$= \lambda_1(w_i)\bar{x}_{i_0} + \sum_{k \in N_i} \frac{\lambda_1(w_k)\lambda_1(w_{ik})}{\lambda_1(w_i)} \bar{x}_{i_0},$$

$$\text{i.e.,} \quad \lambda_1 = \gamma_i + \bar{\gamma}_i, \text{ where } \gamma_i = \lambda_1(w_i) \quad \text{and} \quad \bar{\gamma}_i = \sum_{k \in N_i} \frac{\lambda_1(w_k)\lambda_1(w_{ik})}{\lambda_1(w_i)}.$$

Hence

$$\gamma_1 + \bar{\gamma}_1 = \gamma_2 + \bar{\gamma}_2 = \cdots = \gamma_n + \bar{\gamma}_n.$$

Using Lemma 2.4 we conclude that G is a bipartite regular graph or G is a bipartite semiregular graph.

Conversely, suppose that conditions (i)–(ii) of the theorem hold for the graph G. Let G be a $(\lambda_1(w_{i_0}), \lambda_1(w_{j_0}))$-semiregular bipartite graph. Therefore $\bar{\gamma}_{i_0} = \lambda_1(w_{j_0})$ and $\bar{\gamma}_{j_0} = \lambda_1(w_{i_0})$. Hence

$$\lambda_1 = \lambda_1(w_{i_0}) + \lambda_1(w_{j_0}), \quad \text{by Lemma 2.5},$$

$$\text{i.e.,} \quad \lambda_1 = \max_{ij} \left\{ \frac{\lambda_1(w_i) + \lambda_1(w_j) + \sqrt{(\lambda_1(w_i) - \lambda_1(w_j))^2 + 4\bar{\gamma}_i\bar{\gamma}_j}}{2} \right\}.$$

Hence the theorem is proved. □

Corollary 3.8. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leqslant \max_i \left\{ \lambda_1\left(\sum_{k \in N_i} w_{ik}\right) + \bar{\gamma}_i \right\}, \tag{33}$$

where $\bar{\gamma}_i = \frac{\sum_{k \in N_i} \lambda_1(w_{ik})\lambda_1(w_k)}{\lambda_1(w_i)}$, and w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (33) if and only if

(i) *G is a bipartite regular graph or G is a bipartite semiregular graph; and*
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.*

Proof. For each edge ij we may assume

$$\lambda_1\left(\sum_{k \in N_i} w_{ik}\right) + \bar{\gamma}_i \geqslant \lambda_1\left(\sum_{k \in N_j} w_{jk}\right) + \bar{\gamma}_j,$$

$$\text{i.e.,} \quad \left(\lambda_1\left(\sum_{k \in N_i} w_{ik}\right) - \lambda_1\left(\sum_{k \in N_j} w_{jk}\right)\right) 4\bar{\gamma}_i + 4\bar{\gamma}_i^2 \geqslant 4\bar{\gamma}_i\bar{\gamma}_j,$$

$$\text{i.e.,}\ \lambda_1\left(\sum_{k\in N_i} w_{ik}\right)-\lambda_1\left(\sum_{k\in N_j} w_{jk}\right)+2\bar{\gamma}_i \geqslant \sqrt{\left(\lambda_1\left(\sum_{k\in N_i} w_{ik}\right)-\lambda_1\left(\sum_{k\in N_j} w_{jk}\right)\right)^2+4\bar{\gamma}_i\bar{\gamma}_j},$$

$$\text{i.e.,}\ \lambda_1\left(\sum_{k\in N_i} w_{ik}\right)+\bar{\gamma}_i$$

$$\geqslant \frac{\lambda_1\left(\sum_{k\in N_i} w_{ik}\right)+\lambda_1\left(\sum_{k\in N_j} w_{jk}\right)+\sqrt{\left(\lambda_1\left(\sum_{k\in N_i} w_{ik}\right)-\lambda_1\left(\sum_{k\in N_j} w_{jk}\right)\right)^2+4\bar{\gamma}_i\bar{\gamma}_j}}{2}.$$

Using this result in (24), we get the required result in (33). Using Lemma 2.4 and Theorem 3.7 we get the second part of this Corollary. □

Corollary 3.9. *Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then*

$$\lambda_1 \leqslant \max_i \{w_i + \bar{w}_i\}, \tag{34}$$

where $\bar{w}_i = \frac{\sum_{k\in N_i} w_{ik} w_k}{w_i}$, and w_i is the weight of vertex i. Moreover equality holds in (34) if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.

Proof. We have $\lambda_1(w_i) = w_i$ and $\lambda_1(w_{ij}) = w_{ij}$ for all i, j. Using Corollary 3.8 we get the required result. □

Corollary 3.10 [13]. *Let G be a simple connected unweighted graph. Then*

$$\lambda_1 \leqslant \max_i \{d_i + m_i\}, \tag{35}$$

where d_i is the degree of vertex i and m_i is the average of the degrees of the vertices adjacent to vertex i. Moreover equality holds in (35) if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.

Proof. For undirected graph, $w_{ij} = 1$ for $ij \in E$. Therefore $w_i = d_i$ and $\bar{w}_i = m_i$. Using Corollary 3.9 we get the required result. □

Corollary 3.11. *Let G be a simple connected weighted graph and let each weight w_{ij} be a positive number. Then*

$$\lambda_1 \leqslant \max_{ij}\left\{\frac{w_i + w_j + \sqrt{(w_i - w_j)^2 + 4\bar{w}_i\bar{w}_j}}{2}\right\}, \tag{36}$$

where $\bar{w}_i = \sum_{k\in N_i} \frac{w_k w_{ik}}{w_i}$ and $w_i = \sum_{j\in N_i} w_{ij}$. Moreover equality holds in (36) if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.

Proof. For weighted graph where weight w_{ij} is a positive number, we have $\lambda_1(w_i) = w_i$ and $\lambda_1(w_{ij}) = w_{ij}$ for all i, j. Using Theorem 3.7 we get the required result. □

Corollary 3.12 [4]. *Let G be a simple connected unweighted graph. Then*

$$\lambda_1 \leqslant \max_{ij} \left\{ \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2} \right\}, \tag{37}$$

where d_i is the degree of vertex i and m_i is the average of the degrees of the vertices adjacent to vertex i. Moreover equality holds in (37) if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.

Proof. For unweighted graph, $w_{ij} = 1$ and $w_i = d_i$. Using Corollary 3.11 we get the required result. □

Corollary 3.13 [7]. *Let G be a simple connected weighted graph. Then*

$$\lambda_1 \leqslant \max_{ij} \left\{ \lambda_1 \left(\sum_{k \in N_i} w_{ik} \right) + \sum_{k \in N_j} \lambda_1(w_{jk}) \right\}, \tag{38}$$

where w_{ij} is the positive definite weight matrix of order p of the edge ij. Moreover equality holds in (38) if and only if

(i) *G is a bipartite regular graph or G is a bipartite semiregular graph; and*
(ii) *w_{ij} have a common eigenvector corresponding to the largest eigenvalue $\lambda_1(w_{ij})$ for all i, j.*

Proof. We have

$$\begin{aligned} \lambda_1 &\leqslant \max_{r} \left\{ \lambda_1 \left(\sum_{k \in N_r} w_{rk} \right) + \bar{\gamma}_r \right\}, \quad \text{by (33)} \\ &= \lambda_1 \left(\sum_{k \in N_{i_0}} w_{i_0 k} \right) + \bar{\gamma}_{i_0}, \quad \text{say.} \end{aligned} \tag{39}$$

Let $\lambda_1 \left(\sum_{k \in N_{j_0}} w_{j_0 k} \right) = \max_{k \in N_{i_0}} \lambda_1 \left(\sum_{s \in N_k} w_{ks} \right)$. Then (39) reduces to

$$\begin{aligned} \lambda_1 &\leqslant \lambda_1 \left(\sum_{k \in N_{i_0}} w_{i_0 k} \right) + \lambda_1 \left(\sum_{k \in N_{j_0}} w_{j_0 k} \right) \\ &\leqslant \lambda_1 \left(\sum_{k \in N_{i_0}} w_{i_0 k} \right) + \sum_{k \in N_{j_0}} \lambda_1(w_{j_0 k}), \ \text{as } w_{ij} \text{ is a positive definite matrix} \\ &\leqslant \max_{ij} \left\{ \lambda_1 \left(\sum_{k \in N_i} w_{ik} \right) + \sum_{k \in N_j} \lambda_1(w_{jk}) \right\}. \end{aligned}$$

Using Corollary 3.8 we get the second part of this Corollary. □

Corollary 3.14 [1]. *Let G be a simple connected unweighted graph. Then*

$$\lambda_1 \leqslant \max_{ij}\{d_i + d_j\}, \tag{40}$$

where d_i is the degree of vertex i. Moreover equality holds in (40) if and only if G is a bipartite regular graph or G is a bipartite semiregular graph.

Proof. The proof follows directly from Corollary 3.13. □

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